Radiograph Automata

In completion of Prof. Matthew Cook's course on Models of Computation

Münzel, Lukas Florian
Swiss Federal Institute of Technology in Zürich, Switzerland
lmuenzel@student.ethz.ch

1 Model description

1.1 Formal description

A *radiograph* is defined by a finite set of nodes V and the two sets $E_1, E_2 \subset \{(u, v) \in V \times V\}$, which represent bidirectional edges between the nodes in V s.t. $\forall v \in V : (v, v) \notin E_1 \cup E_2$. The *state* of a radiograph is specified by a partition of V into the sets P, I and O[1]. Nodes in P, I and O are called *passive, in-phase*, and *out-of-phase* respectively. Nodes that are not *passive* are called *active*. This partition is recursively modified in discrete time steps. Specifically, if the partition at time step $t \in \mathbb{N} \cup \{0\}$ is given by P_t, I_t and O_t, then P_{t+1}, I_{t+1} and O_{t+1} are defined by the following rules for every $v \in V$:

1. If no neighbors of v in E_1 are out-of-phase, none in E_2 are in-phase and at least one neighbor in $E_1 \cup E_2$ is active, $v \in I_{t+1}$
2. If no neighbors of v in E_1 are in-phase, none in E_2 are out-of-phase and at least one neighbor in $E_1 \cup E_2$ is active, $v \in O_{t+1}$
3. If $v \notin I_{t+1} \cup O_{t+1}$ (i.e. neither rule one nor rule two apply), then $v \in P_{t+1}$

1.2 Physical analogy

The nodes represent antennas which receive and transmit signals at a fixed frequency. Upon receiving a signal, an antenna always immediately transmits an identical signal at the same frequency to some antennas in its proximity. However, some of the antennas are placed half a wavelength further away then the others. Because the signals received like that are out-of-phase compared to the other incoming signals, destructive interference occurs, and thus no signal is received.

Furthermore, any antenna that only receives out-of-phase signals will again transmit an out-of-phase signal to its neighboring antennas in E_1 and an in-phase signal to its neighboring antennas in E_2. The latter case corresponds to again obtaining the original signal by shifting it by half a wavelength twice.

Please note that this less formal description was included solely to provide a more intuitive understanding of this model. In particular, we want to emphasize the following idealizing assumptions:

1. Even though some signal get shifted by half a wavelength, all simultaneously transmitted signals arrive simultaneously
2. In-phase signals and out-of-phase signals always destructively interfere, resulting in the receiving antenna being passive in the following time step. This is true even if, say, three in-phase signals but only two out-of-phase signal are received.
3. Antennas can be connected arbitrarily as long as no antenna receives a signal directly from itself (i.e. (V, E_1) and (V, E_2) are arbitrary simple graphs)
4. Two antennas might simultaneously be connected by both an edge in E_1 and an edge in E_2.

[1]Meaning that $V = P \sqcup I \sqcup O \iff V = P \cup I \cup O$ and $P \cap I = I \cap O = P \cap O = \varnothing$

1.3 Output and Input

The output of the model is simply represented by the state of specified nodes after an indicated amount of iterations. For example, we might define the output of a model with three possible outputs based on whether a specific antenna is permanently passive, in-phase or out-of-phase after sufficiently many time steps (see section 2 for a more thorough explanation).

The input simply defines the initial state of the radiograph.

2 Example computation

We shall use our model to find which of two binary numbers with some fixed maximum number of bits is larger:

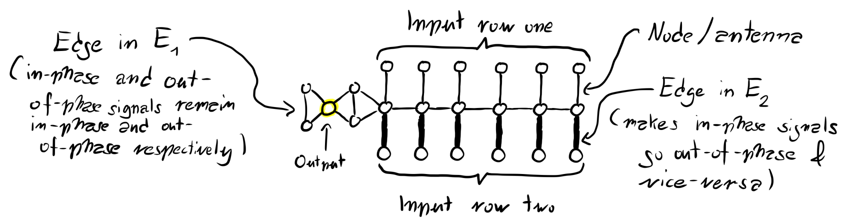


Figure 1: Radiograph to compute the maximum of two binary numbers with at most six bits

To begin the computation, we simply enter the first and second binary number as in-phase antenna activations in the first and second input row respectively with the most significant digit to the left. After sufficiently many iterations, the antenna labeled with output will then constantly be in-phase if the first number was bigger then the second, out-of-phase if the second number was bigger than the first and passive if they were equal. Figure 2 presents a full calculation, where green, red and white nodes represent antennas that are in-phase, out-of-phase and passive respectively and thin lines represent edges in E_1 and thick lines edges in E_2:

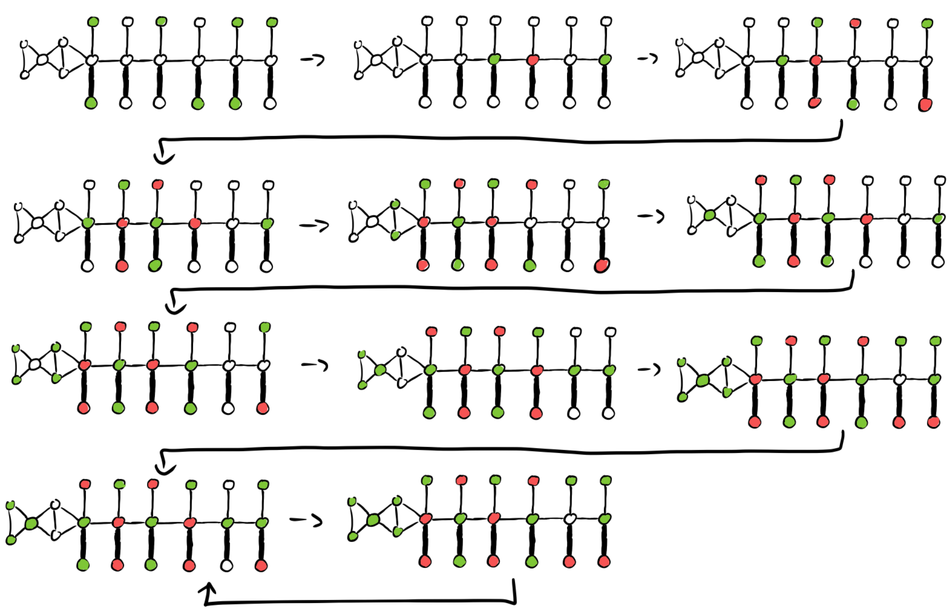


Figure 2: Calculating the maximum of 43 and 38

As desired, because the number 43 is larger than 38, the antenna labeled with *output* in figure 1 is permanently in-phase after sufficiently many time steps.

2.1 Proof why this works

We shall first analyze the leftmost subgraph of six nodes that are not part of the two input rows. In particular, we claim that if the rightmost node of this subgraph (i.e. the node in between the leftmost nodes of the input rows) is in-phase, the output node will permanently be in-phase. We note that only the rightmost node of the subgraph is connected to nodes that are not part of this subgraph. Figure 3 presents the subgraph's state from the initial in-phase activation onwards. Nodes whose state we do not claim to know are visualized as orange and nodes which may either by in-phase or passive as half green, half white. Because we make no assumptions about the state of the neighbors of the rightmost node that are not in the subgraph, the fact that the node corresponding to the output node as defined in figure 1 is eventually permanently in-phase in figure 3 proves our claim.

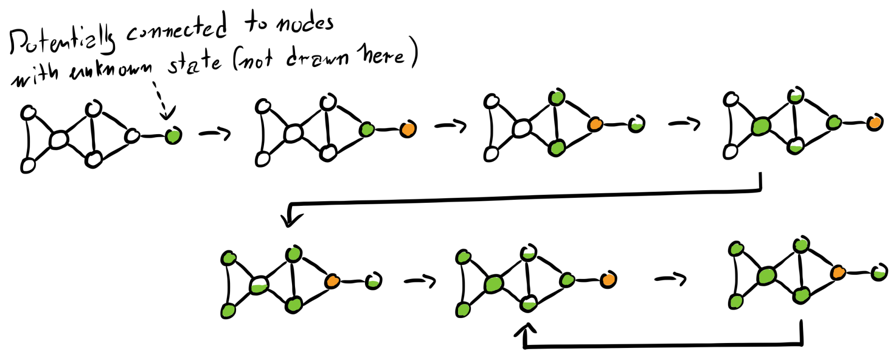


Figure 3: Transitions for radiograph that permanently stores the first activation of the rightmost node

Note that repeating this argument for an initially out-of-phase rightmost node yields an analogous result.

We shall now denote the first number as a and the second as b. We then note that at time step one, the i-th node of the row in between input rows one and two is in-phase if the i-th bit was set in the a but not in b and out-of-phase if it is set in b but not in a. Since the leftmost non-passive signal at time step one will be the first to reach the rightmost node of the subgraph, this signal will eventually be permanently transmitted by the output node. But being the number in which the leftmost bit which is not set in both numbers is set is equivalent to being the larger of two numbers. Thus, the output node must eventually permanently be in-phase if $a > b$ and out-of-phase if $a < b$. If $a = b$, all nodes are passive after the first time step. In particular, the output node is passive, proving that the radiograph does indeed compute the desired result.

3 Comparison to other models

We first note that a given radiograph only has finitely many states because a finite set only admits finitely many partitions. Hence, to simulate any given radiograph, even a cellular automaton with a single cell whose state represents the state of a given radiograph and whose state transitions correspond to the state transitions defined for our radiograph suffices.

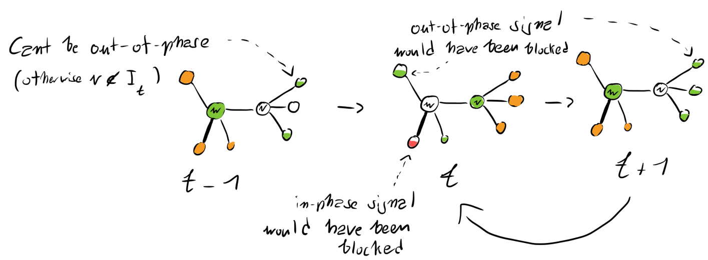
Vice-versa, it's immediate that we can't simulate Turing machines because we only have finitely many states and thus couldn't store arbitrary tapes. Because cellular automata can simulate Turing machines, we therefore also can't simulate arbitrary cellular automata.

So far so good, but what if we consider the more restrictive case of cellular automata with finite grid size? One direction of the argument above remains; the cellular automaton we created to simulate our radiograph only had one cell after all. But can radiographs simulate finite-grid cellular automata? Let us first note that, for example, Conway's game of life can oscillate between three different states for some initial states on a 17x17 grid [2].

We now proof that any given radiograph with any initial condition is eventually periodic with period two:

Assume node v is activated in-phase after $t \geqslant 1$ iterations. Hence, there exists a node w which is only a neighbor of v in E_1 and in-phase after $t-1$ iterations or which is only a neighbor in E_2 and out-of-phase after $t-1$ iterations. In either case, no activation of a neighbor of w could block the in-phase signal from v from again activating w at time step $t+1$, because w's activation at time step $t-1$ would have blocked such an activation of its neighbor at time step t. Hence, the in-phase activation of v at time step t will again activate w at time step $t+1$. But analogously applying this argumentation again while flipping roles of v and w yields that the activation of w at time step $t+1$ will result in an in-phase activation of v at time step $t+2$. Figure 4 illustrates this for the case where w is a neighbor of v in E_1 and thus $w \in I_{t-1}$.

Figure 4: Why activation of v results in another activation of v after two time steps



Note carefully that the above argumentation in writing is independent of whether v is passive, in-phase or out-of-phase at time step $t-1$ and $t+1$. In summary, if $v \in I_t$ for $t \geqslant 1$, $v \in I_{t+2}$. Repeating the above argumentation for $v \in O_t$ also yields that for $t \geqslant 1$, $v \in O_t \implies v \in O_{t+2}$.

Since any given radiograph only has finitely many states, every radiograph with any initial configuration is eventually periodic. So, for a given radiograph, $\exists T, p \in \mathbb{N}$ s.t. $\forall t \geqslant T : P_t = P_{t+p} \wedge I_{t+p} = I_t \wedge O_{t+p} = O_t$. But hence, $\forall t \geqslant T\ \forall v \in V : v \in I_{t+2} \implies I_{t+2+2} \implies \cdots \implies v \in I_{t+2p} = I_{t+p} = I_t$ and combining with the result of the previous paragraph, this yields $\forall t \geqslant T : I_t = I_{t+2}$ and analogously $\forall t \geqslant T : O_t = O_{t+2}$. It follows from $P = O^C \cap P^C$ that $\forall t \geqslant T : P_t = P_{t+2}$ also holds. This completes the proof.

But in order to simulate Game of Life on a 17x17 grid with the cited initial conditions, our radiograph would need be capable of oscillating between at least three different states, which we have just proven to be impossible. Hence, our model is not equivalent to cellular automata with finite grid size.

4 Appendix

Fun fact: Even though we have shown that radiographs are rather weak, we could actually even simulate a Turing machine if we had infinitely many antennas. We only provide a rather rough proof sketch:

1. Implement OR and NOT gates (0 is represented by passive and 1 by in-phase antennas; implementing the NOT requires a small but slightly annoying additional structure)
2. Note that OR and NOT are universal
3. Have rows of infinitely many nodes where the first row represents some initial state of rule 110, the second row the state of rule 110 after the first iteration, etc.
4. Add logic gates between adjacent rows to represent the transition rules for 110
5. Since rule 110 is Turing complete [1], so is this infinite radiograph

Note however that the output format of this infinite radiograph is rather annoying because the states of rule 110 at past time steps might be gradually overwritten by those of the new time steps because radiograph "signals" also flow backwards, at least with the relatively naive implementation of OR and NOT gates I had in mind. Nevertheless, for any $n \in \mathbb{N}$, we find a time step where the n-th row of this infinite radiograph corresponds to the state of rule 110 after simulating a Turing machine for $n-1$ time steps. To me, it therefore seems adequate to call this infinite radiograph Turing complete.

References

[1] Matthew Cook. Universality in elementary cellular automata. *Complex Systems*, 15, 2004.

[2] LifeWiki. Pulsar, 2023. https://conwaylife.com/wiki/Pulsar [Accessed: (30th of July 2023)].